

Maxwell's Equations

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Maxwell's equations are not electromagnetic characteristics. But this is a result derived from the *curl* and the existence of a dual vector field. So, I define dot product of multivectors and *curl* of multivector and will derive Maxwell's equations.

PhySH: classical electromagnetism

I. CURL

A. Dot product of multivectors

$$v_1 \cdot w_1 = (v_i \cdot w_j)$$

$$(v_1 \wedge v_2) \cdot (w_1 \wedge w_2) = \text{determinent of } |(v_i \cdot w_j)| = \begin{vmatrix} (v_1 \cdot w_1) & (v_1 \cdot w_2) \\ (v_2 \cdot w_1) & (v_2 \cdot w_2) \end{vmatrix}$$

$$(v_1 \wedge v_2 \wedge v_3) \cdot (w_1 \wedge w_2 \wedge w_3) = \begin{vmatrix} (v_1 \cdot w_1) & (v_1 \cdot w_2) & (v_1 \cdot w_3) \\ (v_2 \cdot w_1) & (v_2 \cdot w_2) & (v_2 \cdot w_3) \\ (v_3 \cdot w_1) & (v_3 \cdot w_2) & (v_3 \cdot w_3) \end{vmatrix}$$

$$(w_1 \wedge w_2 \wedge v_3) \cdot (v_1 \wedge v_2 \wedge v_3) = (w_1 \wedge w_2 \wedge v_3) \cdot \{-(v_2 \wedge v_1 \wedge v_3)\}$$

$$v_1 \wedge v_2 \wedge v_3 = -(v_2 \wedge v_1 \wedge v_3)$$

$$v_1 \wedge v_1 \wedge v_2 = 0$$

multivector is anticommutative depending on the order of the vectors. Dot product of multivectors is defined in Appendix A.

B. Green theorem and Stokes' theorem

Let's get hints about *curl* from Green's theorem and Stokes' theorem in 2-dimension.

$$\oint (A_1 dx^1 + A_2 dx^2) = \iint \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right) dx^1 dx^2$$

$$\oint A \cdot dl = \iint (\text{curl } A) \cdot ds$$

If u_i is basis and u^i is reciprocal basis at $X(x^1, x^2)$ in a curvilinear coordinates system,

$$u_i = \frac{\partial X}{\partial x^i}, \quad u^i \cdot u_j = \delta_j^i, \quad (i, j, l = 1, 2)$$

in Cartesian coordinates, $e^i \cdot e_i = e_i \cdot e_i = 1$, $e^i = e_i$

$$A_1 dx^1 + A_2 dx^2 =$$

$$(A_1 e^1 + A_2 e^2) \cdot (e_1 dx^1 + e_2 dx^2) =$$

$$(A_1 u^1 + A_2 u^2) \cdot (u_1 dx^1 + u_2 dx^2) =$$

$$A \cdot dl$$

$$A = A_i \mathbb{e}^i, \quad dl = \mathbb{e}_1 dx^1 + \mathbb{e}_2 dx^2, \quad \text{or}$$

$$A = A_i u^i, \quad dl = u_1 dx^1 + u_2 dx^2$$

$$\begin{aligned} & \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right) dx^1 dx^2 = \\ & \left\{ \left(\mathbb{e}^1 \wedge \frac{\partial A_2}{\partial x^1} \mathbb{e}^2 \right) + \left(\mathbb{e}^2 \wedge \frac{\partial A_1}{\partial x^2} \mathbb{e}^1 \right) \right\} \cdot (\mathbb{e}_1 \wedge \mathbb{e}_2) dx^1 dx^2 = \\ & \left\{ \left(u^1 \wedge \frac{\partial A_2}{\partial x^1} u^2 \right) + \left(u^2 \wedge \frac{\partial A_1}{\partial x^2} u^1 \right) \right\} \cdot (u_1 \wedge u_2) dx^1 dx^2 = \\ & \text{curl}(A) \cdot ds \end{aligned}$$

$$\text{curl}(A) = \mathbb{e}^l \wedge \frac{\partial A_i}{\partial x^l} \mathbb{e}^i, \quad ds = (\mathbb{e}_1 \wedge \mathbb{e}_2) dx^1 dx^2, \quad \text{or}$$

$$\text{curl}(A) = u^l \wedge \frac{\partial A_i}{\partial x^l} u^i, \quad ds = (u_1 \wedge u_2) dx^1 dx^2$$

If we describe a vector field A with reciprocal basis in a curvilinear coordinates system, $\text{curl}(A)$ is wedge($\wedge$) product of reciprocal basis u^l and coefficients derivative of A with x^l .

C. Curl in n-dimension

Generalizing in n-dimension, curl of multivector field A is

$$\text{curl}(A) = u^l \wedge \frac{\partial A}{\partial u^l}, \quad (l = 1, \dots, n)$$

$\frac{\partial A}{\partial u^l}$ means coefficients derivative of A

D. $\text{curl}(\text{curl}(A))=0$

$$\text{curl}(\text{curl}(A)) = u^l \wedge u^m \wedge \frac{\partial^2 A}{\partial x^l \partial x^m} = u^m \wedge u^l \wedge \frac{\partial^2 A}{\partial x^m \partial x^l} = 0, \quad (l, m = 1, \dots, n)$$

E. Gradient

$$d\varphi = \frac{\partial \varphi}{\partial x^i} dx^i = \left(u^l \wedge \frac{\partial \varphi}{\partial u^l} \right) \cdot (u_i dx^i) = \text{grad}(\varphi) \cdot dl$$

$$\text{grad}(\varphi) = u^l \wedge \frac{\partial \varphi}{\partial u^l} = \text{curl}(\varphi), \quad (i, l = 1, \dots, n)$$

$\text{gradient}(\varphi)$ is curl of scalar vector field φ . gradient is also curl .

F. Divergence

1. Dual vector in 3-dimension

$$A = A^1 \mathbb{e}_1 + A^2 \mathbb{e}_2 + A^3 \mathbb{e}_3$$

$$A^d = A^1 \mathbb{e}_1^d + A^2 \mathbb{e}_2^d + A^3 \mathbb{e}_3^d = A^1 (\mathbb{e}^2 \wedge \mathbb{e}^3) - A^2 (\mathbb{e}^1 \wedge \mathbb{e}^3) + A^3 (\mathbb{e}^1 \wedge \mathbb{e}^2)$$

A^d is dual vector of A in 3-dimension. dual vector A^d is defined in Appendix B.

2. Divergence in 3-dimension

$$\begin{aligned} \text{div}(A) &= \text{curl}(A^d) = \mathbb{e}^l \wedge \frac{\partial A^d}{\partial x^l} = \\ &= \mathbb{e}^1 \wedge \frac{\partial A^1}{\partial x^1} (\mathbb{e}^2 \wedge \mathbb{e}^3) - \mathbb{e}^2 \wedge \frac{\partial A^2}{\partial x^2} (\mathbb{e}^1 \wedge \mathbb{e}^3) + \mathbb{e}^3 \wedge \frac{\partial A^3}{\partial x^3} (\mathbb{e}^1 \wedge \mathbb{e}^2) = \\ &= \left(\frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} \right) (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) \end{aligned}$$

In other words, $\text{divergence}(A)$ is curl of dual vector field A^d . divergence is also curl .

G. Examples in 3-dimension spherical coordinates

Converting conventional spherical basis $\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\phi$ to curvilinear basis u_r, u_θ, u_ϕ

$$\begin{aligned} x^k &= r, \theta, \phi \\ u_r &= \frac{\partial X}{\partial r} = \mathbb{e}_r, \quad u_\theta = \frac{\partial X}{\partial \theta} = r \mathbb{e}_\theta, \quad u_\phi = \frac{\partial X}{\partial \phi} = r \sin \theta \mathbb{e}_\phi \\ u^r &= \mathbb{e}_r, \quad u^\theta = \frac{\mathbb{e}_\theta}{r}, \quad u^\phi = \frac{\mathbb{e}_\phi}{r \sin \theta} \end{aligned}$$

$$A = A_r \mathbb{e}_r + A_\theta \mathbb{e}_\theta + A_\phi \mathbb{e}_\phi = A_r u^r + A_\theta r u^\theta + A_\phi r \sin \theta u^\phi$$

$$A^d = A_r (\mathbb{e}_\theta \wedge \mathbb{e}_\phi) - A_\theta (\mathbb{e}_r \wedge \mathbb{e}_\phi) + A_\phi (\mathbb{e}_r \wedge \mathbb{e}_\theta) =$$

$$A_r r^2 \sin \theta (u^\theta \wedge u^\phi) - A_\theta r \sin \theta (u^r \wedge u^\phi) + A_\phi r (u^r \wedge u^\theta)$$

Dual vector A^d is defined in Appendix B.

1. Curl

Let's calculate $(u^\theta \wedge u^\phi)$ term of $\text{curl}(A)$.

$$\begin{aligned} u^\theta \wedge \frac{\partial}{\partial \theta} (A_\phi r \sin \theta) u^\phi + u^\phi \wedge \frac{\partial}{\partial \phi} (A_\theta r) u^\theta &= \left\{ \frac{\partial}{\partial \theta} (A_\phi r \sin \theta) - \frac{\partial}{\partial \phi} (A_\theta r) \right\} (u^\theta \wedge u^\phi) = \\ &= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial \theta} (A_\phi r \sin \theta) - \frac{\partial}{\partial \phi} (A_\theta r) \right\} (\mathbb{e}_\theta \wedge \mathbb{e}_\phi) \end{aligned}$$

The coefficient of $(\mathbb{e}_\theta \wedge \mathbb{e}_\phi)$ term and the coefficient of $\mathbb{e}_r$ term in conventional expression are the same. We are mistaking $(\mathbb{e}_\theta \wedge \mathbb{e}_\phi)$ term for $\mathbb{e}_r$ term.

2. Gradient

$$\begin{aligned} \text{grad}(\varphi) &= \text{curl}(\varphi) = u^l \wedge \frac{\partial \varphi}{\partial u^l} = \\ u^r \frac{\partial \varphi}{\partial r} + u^\theta \frac{\partial \varphi}{\partial \theta} + u^\phi \frac{\partial \varphi}{\partial \phi} &= \frac{\partial \varphi}{\partial r} \mathbb{e}_r + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \mathbb{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \phi} \mathbb{e}_\phi \end{aligned}$$

3. Divergence

$$\text{div}(A) = \text{curl}(A^d) = u^l \wedge \frac{\partial A^d}{\partial x^l} =$$

$$\left\{ \frac{\partial}{\partial r} (A_r r^2 \sin \theta) + \frac{\partial}{\partial \theta} (A_\theta r \sin \theta) + \frac{\partial}{\partial \phi} (A_\phi r) \right\} (u^r \wedge u^\theta \wedge u^\phi) =$$

$$\frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (A_r r^2 \sin \theta) + \frac{\partial}{\partial \theta} (A_\theta r \sin \theta) + \frac{\partial}{\partial \phi} (A_\phi r) \right\} (\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\phi)$$

The coefficient of $(\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\phi)$ term is the same $div(A)$ in conventional expression, but we are mistaking $(\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\phi)$ term for scalar term.

II. MAXWELL'S EQUATIONS

A. Magnetic field and electric field

We prove in 4-dimension Cartesian coordinates.

$$\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_t$$

$$\mathbb{e}^t \cdot \mathbb{e}_t = 1 = -\mathbb{e}_t \cdot \mathbb{e}_t, \quad \mathbb{e}^t = -\mathbb{e}_t$$

If there is a vector field (vector potential) A in 4-dim, then $curl(A)$ makes unique 2-vector field, magnetic field $\mathcal{B}$ and electric field $\mathcal{E}$. $curl(A) = \mathcal{B} + \mathcal{E}$.

$$A = A_j \mathbb{e}^j + \varphi \mathbb{e}^t \quad (i, j, k, l, m = 1, 2, 3)$$

$$curl(A) = \mathbb{e}^i \wedge \frac{\partial A}{\partial x^i} + \mathbb{e}^t \wedge \frac{\partial A}{\partial t} =$$

$$\frac{\partial A_j}{\partial x^i} (\mathbb{e}^i \wedge \mathbb{e}^j) + \left\{ \frac{\partial \varphi}{\partial x^k} (\mathbb{e}^k \wedge \mathbb{e}^t) + \frac{\partial A_k}{\partial t} (\mathbb{e}^t \wedge \mathbb{e}^k) \right\} = \mathcal{B} + \mathcal{E}$$

$$\mathcal{B} = \frac{\partial A_j}{\partial x^i} (\mathbb{e}^i \wedge \mathbb{e}^j), \quad (i, j \text{ sum})$$

$$\mathcal{E} = \left(\frac{\partial \varphi}{\partial x^k} - \frac{\partial A^k}{\partial t} \right) (\mathbb{e}^k \wedge \mathbb{e}^t)$$

But if we express two terms $(\mathbb{e}^i \wedge \mathbb{e}^j)$ ($i, j \text{ sum}$) as one term $(\mathbb{e}^i \wedge \mathbb{e}^j)$, ($i < j$).

$$\mathcal{B} = \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) (\mathbb{e}^i \wedge \mathbb{e}^j) = \varepsilon^{kij} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) \varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j) =$$

$$B^k \varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j), \quad (i < j)$$

$$B^k = \varepsilon^{kij} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right), \quad (i < j)$$

$\varepsilon^{kij}, \varepsilon_{kij}$ are Levi-Civita symbol.

$$\mathcal{E} = \left(\frac{\partial \varphi}{\partial x^k} - \frac{\partial A^k}{\partial t} \right) (\mathbb{e}^k \wedge \mathbb{e}^t) = E_k (\mathbb{e}^k \wedge \mathbb{e}^t)$$

$$E_k = \frac{\partial \varphi}{\partial x^k} - \frac{\partial A^k}{\partial t}$$

Conventional magnetic field B and electric field E .

$$B = B^k \mathbb{e}_k, \quad E = E_k \mathbb{e}^k$$

B. Gauss's law for magnetism, Faraday's law

3-vector field, $\text{curl}(\text{curl}(A)) = 0$ is Gauss's law for magnetism and Faraday's law. That is not characteristics of electromagnetism, but curl .

$$\begin{aligned} \text{curl}(\text{curl}(A)) &= \text{curl}(\mathcal{B} + \mathcal{E}) = \left(\mathbb{e}^k \wedge \frac{\partial \mathcal{B}}{\partial x^k} \right) + \left\{ \left(\mathbb{e}^t \wedge \frac{\partial \mathcal{B}}{\partial t} \right) + \left(\mathbb{e}^l \wedge \frac{\partial \mathcal{E}}{\partial x^l} \right) \right\} = \\ &= \frac{\partial B^k}{\partial x^k} \varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j) + \left\{ \frac{\partial B^k}{\partial t} \varepsilon_{kij} (\mathbb{e}^t \wedge \mathbb{e}^i \wedge \mathbb{e}^j) + \frac{\partial E_m}{\partial x^l} (\mathbb{e}^l \wedge \mathbb{e}^m \wedge \mathbb{e}^t) \right\} = 0, \quad (i < j) \end{aligned}$$

If we sort terms, each term must be 0. and $(\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j)$ terms and $(\mathbb{e}^t \wedge \mathbb{e}^i \wedge \mathbb{e}^j)$ terms can be expressed dual basis $\mathbb{e}_k^d, \mathbb{e}_t^d$. They are defined in Appendix B.

$$\begin{aligned} \varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j) &= \mathbb{e}_t^d = \mathbb{e}_t \cdot I \\ \varepsilon_{klm} (\mathbb{e}^l \wedge \mathbb{e}^m \wedge \mathbb{e}^t) &= -\mathbb{e}_k^d = -\mathbb{e}_k \cdot I \\ I &= \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 \wedge \mathbb{e}_t = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t \end{aligned}$$

The first term is Gauss's law for magnetism.

$$\frac{\partial B^k}{\partial x^k} \varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j) = (\nabla \cdot \mathcal{B}) \mathbb{e}_t \cdot I = 0$$

The term in parentheses is Faraday's law.

$$\begin{aligned} &\frac{\partial B^k}{\partial t} \varepsilon_{kij} (\mathbb{e}^t \wedge \mathbb{e}^i \wedge \mathbb{e}^j) + \frac{\partial E_m}{\partial x^l} (\mathbb{e}^l \wedge \mathbb{e}^m \wedge \mathbb{e}^t) = \\ &= - \left(\frac{\partial B^k}{\partial t} + \varepsilon^{klm} \frac{\partial E_m}{\partial x^l} \right) \mathbb{e}_k \cdot I = \\ &= - \left(\frac{\partial \mathcal{B}}{\partial t} + \nabla \times \mathcal{E} \right) \cdot I = 0 \end{aligned}$$

C. Dual magnetic field and dual electric field

Also, there are unique dual magnetic field $\mathcal{B}^d$ and dual electric field $\mathcal{E}^d$ corresponding to $\mathcal{B}, \mathcal{E}$.

$$\begin{aligned} \mathcal{B}^d + \mathcal{E}^d &= B^k \varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j)^d + E_k (\mathbb{e}^k \wedge \mathbb{e}^t)^d = \\ &= -B^k (\mathbb{e}^k \wedge \mathbb{e}^t) + E_k \varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j), \quad (i < j) \end{aligned}$$

Dual basis $(\mathbb{e}^i \wedge \mathbb{e}^j)^d, (\mathbb{e}^k \wedge \mathbb{e}^t)^d$ is defined in Appendix B.

D. Ampere's law, Gauss's law

And $\text{curl}(\mathcal{B}^d + \mathcal{E}^d)$ makes 3-vector field $\mu \mathcal{J}$. That is Ampere's law and Gauss's law.

$$\begin{aligned} \text{curl}(\mathcal{B}^d + \mathcal{E}^d) &= \mu \mathcal{J} \\ \mathcal{J} &= J^k \mathbb{e}_k^d + \rho \mathbb{e}_t^d = (J^k \mathbb{e}_k + \rho \mathbb{e}_t) \cdot I = (J + \rho \mathbb{e}_t) \cdot I, \quad J = J^k \mathbb{e}_k \\ &\left\{ -\frac{\partial B^m}{\partial x^l} (\mathbb{e}^l \wedge \mathbb{e}^m \wedge \mathbb{e}^t) + \varepsilon_{kij} \frac{\partial E_k}{\partial t} (\mathbb{e}^t \wedge \mathbb{e}^i \wedge \mathbb{e}^j) \right\} + \varepsilon_{kij} \frac{\partial E_k}{\partial x^k} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j) = \end{aligned}$$

$$\left(\varepsilon_{klm} \frac{\partial B^m}{\partial x^l} - \frac{\partial E_k}{\partial t}\right) \mathbb{e}_k \cdot I + \left(\frac{\partial E_k}{\partial x^k}\right) \mathbb{e}_t \cdot I = \mu(J^k \mathbb{e}_k + \rho \mathbb{e}_t) \cdot I$$

$\mathbb{e}_k \cdot I$ term is Ampere's law.

$$\nabla \times B = \frac{\partial E}{\partial t} + \mu J \rightarrow \nabla \times H = \frac{\partial D}{\partial t} + J, \quad \frac{B}{\mu} = H, \quad \frac{E}{\mu} = D$$

$\mathbb{e}_t \cdot I$ term is Gauss's law.

$$\nabla \cdot E = \mu \rho, \quad \nabla \cdot D = \rho$$

E. charge conservation

4-vector field $\text{curl}(B^d + E^d) = \text{curl}(\mu J) = 0$ is conservative law. That is not characteristics of nature, but curl .

$$\text{curl}(J) = \frac{\partial J^k}{\partial x^k} (\mathbb{e}^k \wedge \mathbb{e}_k^d) + \frac{\partial \rho}{\partial t} (\mathbb{e}^t \wedge \mathbb{e}_t^d) = \left(\nabla \cdot J + \frac{\partial \rho}{\partial t}\right) I = 0$$

$\mathbb{e}^k \wedge \mathbb{e}_k^d = \mathbb{e}^t \wedge \mathbb{e}_t^d = I$ is explained in Appendix B.

APPENDIX A

Dot product of multivector

$$a \cdot (v_1 \wedge \dots \wedge v_r) = (v_1 \wedge \dots \wedge v_r) \cdot a =$$

$$(-1)^{i-1} (a \cdot v_i) (v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge v_r) \quad \text{Einstein summation}$$

$$v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge v_r = v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r$$

$$(w_1 \wedge \dots \wedge w_q) \cdot (v_1 \wedge \dots \wedge v_r) =$$

$$w_q \cdot (w_{q-1} \cdot \dots \cdot w_2 \cdot (w_1 \cdot (v_1 \wedge \dots \wedge v_r)) \dots) =$$

$$(w_2 \wedge \dots \wedge w_q) \cdot (w_1 \cdot (v_1 \wedge \dots \wedge v_r)) = \dots, \quad (q \leq r)$$

$$(a \wedge b) \cdot (v_1 \wedge \dots \wedge v_r) = (v_1 \wedge \dots \wedge v_r) \cdot (a \wedge b) = b \cdot \{a \cdot (v_1 \wedge \dots \wedge v_r)\}$$

Calculating of $(a \cdot v_i)(b \cdot v_j)$ term, $(i < j)$.

$$b \cdot \{a \cdot (v_1 \wedge \dots \wedge v_r)\}$$

$$(-1)^{i-1} (a \cdot v_i) \{b \cdot (v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge v_r)\} =$$

$$(-1)^{i-1} (a \cdot v_i) (-1)^j (b \cdot v_j) (v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge \check{v}_j \wedge \dots \wedge v_r) =$$

$$-(-1)^{j-1} (b \cdot v_j) (-1)^{i-1} (a \cdot v_i) (v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge \check{v}_j \wedge \dots \wedge v_r) =$$

$$-a \cdot \{b \cdot (v_1 \wedge \dots \wedge v_r)\} =$$

$$-(b \wedge a) \cdot (v_1 \wedge \dots \wedge v_r) =$$

$$-(a \wedge b) \cdot (v_1 \wedge \dots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \dots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \dots \wedge v_r)$$

$$a \wedge b = -b \wedge a$$

$$v_1 \wedge \dots \wedge v_r = -v_1 \wedge \dots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \dots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \dots \wedge v_r$$

multivector is anticommutative depending on the order of the vector.

$$(w_1 \wedge \dots \wedge w_r) \cdot (v_1 \wedge \dots \wedge v_r) = (v_1 \wedge \dots \wedge v_r) \cdot (w_1 \wedge \dots \wedge w_r) = |(w_i \cdot v_j)|$$

APPENDIX B

Pseudoscalar I in 3-Dim, dual basis, and dual vector in 3-Dim

$$\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \quad (i, j, k = 1, 2, 3)$$

$$\mathbb{e}^i \cdot \mathbb{e}_j = \mathbb{e}_i \cdot \mathbb{e}_j, \quad \mathbb{e}^i = \mathbb{e}_i$$

$$I = \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 = \mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 I \neq u_1 \wedge u_2 \wedge u_3$$

$$\mathbb{e}_1^d = \mathbb{e}_1 \cdot I = \mathbb{e}_1 \cdot (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) = \mathbb{e}^2 \wedge \mathbb{e}^3$$

$$\mathbb{e}_2^d = -\mathbb{e}^1 \wedge \mathbb{e}^3, \quad \mathbb{e}_3^d = \mathbb{e}^1 \wedge \mathbb{e}^2$$

$$\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\phi = \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 = I$$

$$\mathbb{e}_r^d = (\mathbb{e}_\theta \wedge \mathbb{e}_\phi), \quad \mathbb{e}_\theta^d = -(\mathbb{e}_r \wedge \mathbb{e}_\phi), \quad \mathbb{e}_\phi^d = (\mathbb{e}_r \wedge \mathbb{e}_\theta)$$

$$A = A^1 \mathbb{e}_1 + A^2 \mathbb{e}_2 + A^3 \mathbb{e}_3$$

$$A^d = A \cdot I = A^1 \mathbb{e}_1^d + A^2 \mathbb{e}_2^d + A^3 \mathbb{e}_3^d = A^1 (\mathbb{e}^2 \wedge \mathbb{e}^3) - A^2 (\mathbb{e}^1 \wedge \mathbb{e}^3) + A^3 (\mathbb{e}^1 \wedge \mathbb{e}^2)$$

Pseudoscalar I in 4-Dim, dual basis, and dual vector in 4-Dim

$$\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_t$$

$$\mathbb{e}^i = \mathbb{e}_i, \quad \mathbb{e}^t \cdot \mathbb{e}_t = 1 = -\mathbb{e}_t \cdot \mathbb{e}_t, \quad \mathbb{e}^t = -\mathbb{e}_t$$

$$I = \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 \wedge \mathbb{e}_t = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t$$

$$\mathbb{e}_k^d = \mathbb{e}_k \cdot I = -\mathbb{e}_k \cdot (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) =$$

$$-\mathbb{e}_k \cdot \varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t) = -\varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t)$$

$$(\mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t) = -\varepsilon_{kij} \mathbb{e}_k^d$$

$$\mathbb{e}^k \wedge \mathbb{e}_k^d = -\varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t) = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t = I$$

$$\mathbb{e}_t^d = \mathbb{e}_t \cdot I = -\mathbb{e}_t \cdot (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$\mathbb{e}^t \wedge \mathbb{e}_t^d = \mathbb{e}^t \wedge \mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t = I$$

$$(\mathbb{e}^i \wedge \mathbb{e}^j)^d = (\mathbb{e}^i \wedge \mathbb{e}^j) \cdot \varepsilon_{kij} (\mathbb{e}_i \wedge \mathbb{e}_j \wedge \mathbb{e}_k \wedge \mathbb{e}_t) = \varepsilon_{kij} (\mathbb{e}_k \wedge \mathbb{e}_t) = -\varepsilon_{kij} (\mathbb{e}^k \wedge \mathbb{e}^t)$$

$$(\mathbb{e}^k \wedge \mathbb{e}^t)^d = (\mathbb{e}^k \wedge \mathbb{e}^t) \cdot \varepsilon_{kij} (\mathbb{e}_i \wedge \mathbb{e}_j \wedge \mathbb{e}_k \wedge \mathbb{e}_t) = \varepsilon_{kij} (\mathbb{e}_i \wedge \mathbb{e}_j) = \varepsilon_{kij} (\mathbb{e}^i \wedge \mathbb{e}^j)$$